A family of (2+1)-dimensional hydrodynamic type systems possessing pseudopotential

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Abstract

We construct a family of integrable hydrodynamic type systems with three independent and $n \geq 2$ dependent variables in terms of solutions of linear system of PDEs with rational coefficients. We choose the existence of a pseudopotential as a criterion of integrability. In the case $n = 2$ this family is a general solution of the classification problem for such systems. We give also an elliptic analog of this family in the case $n > 2$.

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Introduction

We address the problem of classification of integrable (2+1)-dimensional quasilinear systems

$$\mathbf{u}_t = A(\mathbf{u})\,\mathbf{u}_x + B(\mathbf{u})\,\mathbf{u}_y, \tag{0.1}$$

where t, x, y are independent variables, $\mathbf{u}$ is an n-component column vector, and $A(\mathbf{u})$ and $B(\mathbf{u})$ are $n \times n$-matrices. A general theory of such systems was developed in the papers [1, 2, 3]. This theory is based on the existence of sufficiently many of the hydrodynamic reductions [4, 1] which has been proposed as the definition of integrability. In the first nontrivial case $n = 2$ the complete set of integrability conditions has been found in the paper [2] in the form of a complicated system of PDEs for the entries of the matrices A and B. In the case $n > 2$ the complete set of integrability conditions in terms of the matrices A and B is not known. However, all known examples of integrable (2+1)-dimensional hydrodynamic type systems possess the so-called scalar pseudopotential[1]

$$\Psi_t = f(\Psi_y, \mathbf{u}), \qquad \Psi_x = g(\Psi_y, \mathbf{u}). \tag{0.2}$$

Moreover, it was proven in [2] that for $n = 2$ the integrability conditions are equivalent to the existence of the scalar pseudopotential. The scalar pseudopotential plays an important role in the theory of the universal Whitham hierarchy [5, 6, 7]. Various integrable systems possessing pseudopotential were constructed and studied in [8], [9] and many other papers. In the paper [10] the case of a constant matrix A was considered and hydrodynamic type systems (0.1) possessing a pseudopotential with movable singularities were classified.

In this paper we give the general solution to the classification problem for the integrable hydrodynamic type systems (0.1) in the case $n = 2$. It turns out that the answer can be written in terms of three arbitrary linear independent solutions of a certain system of linear PDEs with rational coefficients and a 3-dimensional space of solutions. The action of the group GL_3 on this space corresponds to the action of the same group on the space spanned by the independent variables t, x, y. It is also possible to construct a similar family of hydrodynamic type systems possessing a pseudopotential for higher n. The system is written in terms of three arbitrary linear independent solutions of a certain system of linear PDEs with rational coefficients and an $n+1$-dimensional space of solutions.

Let us describe the contents of the paper. In Section 1 we recall the background material. In Section 2 we construct a family of hydrodynamic type systems possessing a pseudopotential for arbitrary $n \geq 2$. In Section 3 we recall the results of the paper [2] on two components integrable systems and prove that our family in the case $n = 2$ is a general solution for the classification problem. In Section 4 we give an elliptic analog of our construction.

[1]This means that the overdetermined system (0.2) for Ψ is compatible if and only if $\mathbf{u}$ is a solution of (0.1).

1 Hydrodynamic type systems and their pseudopotentials

Consider a $(2+1)$-dimensional quasilinear system

$$u_{it} = \sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{jx} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{jy}, \quad i = 1, ..., n. \tag{1.3}$$

This system is called integrable [1] if it possesses ‘sufficiently many’ exact solutions of the form $\mathbf{u} = \mathbf{u}(R^1, ..., R^N)$ where the so-called Riemann invariants $R^1, ..., R^N$ solve a pair of commuting diagonal systems

$$R^i_t = \lambda^i(R)\ R^i_y, \qquad R^i_x = \mu^i(R)\ R^i_y, \tag{1.4}$$

and the number N of Riemann invariants is allowed to be arbitrary.

A pair of equations of the form

$$\Psi_t = f(\Psi_y, u_1, \dots, u_n), \qquad \Psi_x = g(\Psi_y, u_1, \dots, u_n), \tag{1.5}$$

with respect to unknown Ψ is called a pseudopotential for equation (1.3) if the compatibility condition $\Psi_{tx} = \Psi_{xt}$ for (1.5) is equivalent to (1.3). Differentiating (1.5), we find that this compatibility condition is given by

$$f_\xi \sum_{i=1}^{n} u_{iy}\, g_{u_i} + \sum_{i=1}^{n} u_{ix}\, f_{u_i} = g_\xi \sum_{i=1}^{n} u_{iy}\, f_{u_i} + \sum_{i=1}^{n} u_{it}\, g_{u_i}. \tag{1.6}$$

Here and below we denote Ψ_y by ξ. Substituting the right hand side of (1.3) for t-derivatives and splitting with respect to x and y-derivatives, we get that for any i the following relations hold:

$$f_{u_i} = \sum_{j=1}^{n} a_{ji} g_{u_j}, \tag{1.7}$$

$$f_\xi\, g_{u_i} - g_\xi\, f_{u_i} = \sum_{j=1}^{n} b_{ji} g_{u_j}. \tag{1.8}$$

We will use the following

Lemma 1. A pair of equations (1.5) is a pseudopotential for some system of the form (1.3) iff the functions $\{f_{u_i}, g_{u_i}, f_\xi\, g_{u_i} - g_\xi f_{u_i}; i = 1, ..., n\}$ constitute an n-dimensional linear subspace in the space of functions in ξ and $\{g_{u_i}; i = 1, ..., n\}$ is a basis in this space.

Proof. Indeed, $\{g_{u_i}; i = 1, ..., n\}$ is a basis in this space of functions in ξ iff $f_{u_i}, f_\xi\, g_{u_i} - g_\xi f_{u_i}$ can be uniquely written in the form (1.7), (1.8) where coefficients a_{ji}, b_{ji} does not depend on ξ.

In some cases it is more convenient to define a pseudopotential in parametric form

$$\Psi_y = F(\zeta, u_1, \dots, u_n), \qquad \Psi_t = G(\zeta, u_1, \dots, u_n), \qquad \Psi_x = H(\zeta, u_1, \dots, u_n) \tag{1.9}$$

which takes the form (1.5) if one expresses the parameter ζ in terms of Ψ_y from the first equation and substitutes into the second and the third equations. The form (1.9) is more symmetric with respect to the variables t, x, y. In particular, linear change of the variables y, t, x corresponds to the same linear change of the functions F, G, H. The compatibility conditions for the system (1.9) read

$$\sum_{i=1}^{n}(H_\zeta G_{u_i} - G_\zeta H_{u_i})u_{iy} + \sum_{i=1}^{n}(F_\zeta H_{u_i} - H_\zeta F_{u_i})u_{it} + \sum_{i=1}^{n}(G_\zeta F_{u_i} - F_\zeta G_{u_i})u_{ix} = 0. \tag{1.10}$$

We can slightly generalize Lemma 1 in the following way:

Lemma 2. The compatibility conditions for the system (1.9) are equivalent to a quasilinear system of m linear independent equations the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{jy} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{jt} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{jx} = 0, \quad i = 1, ..., m \tag{1.11}$$

iff the functions $\{H_\zeta G_{u_i} - G_\zeta H_{u_i}, F_\zeta H_{u_i} - H_\zeta F_{u_i}, G_\zeta F_{u_i} - F_\zeta G_{u_i}; i = 1, ..., n\}$ constitute an m-dimensional linear subspace in the space of functions in ζ.

Proof. Let $\{S_1(\zeta), ..., S_m(\zeta)\}$ be a basis in this linear space and

$$H_\zeta G_{u_i} - G_\zeta H_{u_i} = \sum_{j=1}^{m} a_{ji}(\mathbf{u}) S_j, \quad F_\zeta H_{u_i} - H_\zeta F_{u_i} = \sum_{j=1}^{m} b_{ji}(\mathbf{u}) S_j, \quad G_\zeta F_{u_i} - F_\zeta G_{u_i} = \sum_{j=1}^{m} c_{ji}(\mathbf{u}) S_j.$$

Substituting these equations into (1.10) and equating to zero coefficients at $S_1, ..., S_m$ we obtain (1.11).

2 Construction of a family of n-component systems possessing pseudopotential

Define a function $F(\zeta, u_1, \ldots, u_n)$ as a solution of the following systems of PDEs

$$F_\zeta = \phi(\zeta) \cdot \zeta^{-s_1}(\zeta - 1)^{-s_2}(\zeta - u_1)^{-s_3} ... (\zeta - u_n)^{-s_{n+2}},$$

$$F_{u_i} = -\frac{\phi(u_i)}{\zeta - u_i} \cdot \frac{\zeta^{1-s_1}(\zeta - 1)^{1-s_2}(\zeta - u_1)^{1-s_3} ... (\zeta - u_n)^{1-s_{n+2}}}{u_i(u_i - 1)(u_i - u_1) ... \hat{i} ... (u_i - u_n)}, \quad i = 1, ..., n. \tag{2.12}$$

Here $s_1, ..., s_{n+2}$ are constants and

$$\phi(\zeta) = \alpha_0(u_1, ..., u_n) + \alpha_1(u_1, ..., u_n)\zeta + ... + \alpha_n(u_1, ..., u_n)\zeta^n$$

is a polynomial of degree n. The notation $\hat{i}$ means that the ith multiplier is omitted in the product. The system (2.12) is in involution iff the polynomial ϕ satisfies the following system of PDEs

$$\phi_{u_i}(\zeta) = \phi(u_i) \frac{\zeta(\zeta - 1)(\zeta - u_1) ... \hat{i} ... (\zeta - u_n)}{u_i(u_i - 1)(u_i - u_1) ... \hat{i} ... (u_i - u_n)} \times \tag{2.13}$$

$$\left(\frac{s_1-1}{\zeta}+\frac{s_2-1}{\zeta-1}+\frac{s_3-1}{\zeta-u_1}+...+\frac{s_{i+2}}{\zeta-u_i}+...+\frac{s_{n+2}-1}{\zeta-u_n}\right)-\frac{s_{i+2}}{\zeta-u_i}\phi(\zeta), \quad i=1,...,n.$$

Here $\phi_{u_i}(\zeta)=\alpha_{0u_i}+\alpha_{1u_i}\zeta+...+\alpha_{nu_i}\zeta^n$. It is clear that if ϕ is a polynomial of degree n, then the right hand side of (2.13) is also a polynomial of degree n. Therefore, (2.13) is a well-defined system of linear PDEs for coefficients of ϕ. It can be checked straightforwardly that this system is in involution. Therefore, there are $n+1$ linear independent solutions.

Let ϕ, ϕ_1 and ϕ_2 be three linear independent solutions of the system (2.13). We assume that ϕ, ϕ_1, ϕ_2 are polynomials of degree n with respect to ζ. Define functions $G(\zeta,u_1,\dots,u_n)$ and $H(\zeta,u_1,\dots,u_n)$ similarly to (2.12) by

$$G_\zeta=\phi_1(\zeta)\cdot\zeta^{-s_1}(\zeta-1)^{-s_2}(\zeta-u_1)^{-s_3}...(\zeta-u_n)^{-s_{n+2}},$$

$$G_{u_i}=-\frac{\phi_1(u_i)}{\zeta-u_i}\cdot\frac{\zeta^{1-s_1}(\zeta-1)^{1-s_2}(\zeta-u_1)^{1-s_3}...(\zeta-u_n)^{1-s_{n+2}}}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}, \quad i=1,...,n \tag{2.14}$$

for the function G and

$$H_\zeta=\phi_2(\zeta)\cdot\zeta^{-s_1}(\zeta-1)^{-s_2}(\zeta-u_1)^{-s_3}...(\zeta-u_n)^{-s_{n+2}},$$

$$H_{u_i}=-\frac{\phi_2(u_i)}{\zeta-u_i}\cdot\frac{\zeta^{1-s_1}(\zeta-1)^{1-s_2}(\zeta-u_1)^{1-s_3}...(\zeta-u_n)^{1-s_{n+2}}}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}, \quad i=1,...,n \tag{2.15}$$

for the function H.

Proposition 1. If the functions F,G,H are defined by (2.12), (2.14) and (2.15), then the system (1.9) defines a pseudopotential for some system of the form (1.11) with $m=n$.

Proof. Equations (2.12), (2.14), (2.15) imply

$$\begin{gathered}
H_\zeta G_{u_i}-G_\zeta H_{u_i}=\vartheta_i(\zeta)\cdot\zeta^{1-2s_1}(\zeta-1)^{1-2s_2}(\zeta-u_1)^{1-2s_3}...(\zeta-u_n)^{1-2s_{n+2}},\\
F_\zeta H_{u_i}-H_\zeta F_{u_i}=\nu_i(\zeta)\cdot\zeta^{1-2s_1}(\zeta-1)^{1-2s_2}(\zeta-u_1)^{1-2s_3}...(\zeta-u_n)^{1-2s_{n+2}},\\
G_\zeta F_{u_i}-F_\zeta G_{u_i}=\mu_i(\zeta)\cdot\zeta^{1-2s_1}(\zeta-1)^{1-2s_2}(\zeta-u_1)^{1-2s_3}...(\zeta-u_n)^{1-2s_{n+2}}
\end{gathered} \tag{2.16}$$

where functions $\vartheta_i(\zeta),\nu_i(\zeta),\mu_i(\zeta)$ are defined by

$$\begin{aligned}
\vartheta_i(\zeta)&=\frac{1}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}\cdot\frac{\phi_1(u_i)\phi_2(\zeta)-\phi_2(u_i)\phi_1(\zeta)}{\zeta-u_i},\\
\nu_i(\zeta)&=\frac{1}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}\cdot\frac{\phi_2(u_i)\phi(\zeta)-\phi(u_i)\phi_2(\zeta)}{\zeta-u_i},\\
\mu_i(\zeta)&=\frac{1}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}\cdot\frac{\phi(u_i)\phi_1(\zeta)-\phi_1(u_i)\phi(\zeta)}{\zeta-u_i}.
\end{aligned} \tag{2.17}$$

It is clear that $\vartheta_i(\zeta)$ $\nu_i(\zeta)$, $\mu_i(\zeta)$ are polynomials in the variable ζ of degree $n-1$. Therefore, the space of functions $\{H_\zeta G_{u_i}-G_\zeta H_{u_i}, F_\zeta H_{u_i}-H_\zeta F_{u_i}, G_\zeta F_{u_i}-F_\zeta G_{u_i}; i=1,...,n\}$ in the variable

ζ is isomorphic to the space of polynomials in ζ of degree less or equal to $n-1$. This space is n-dimensional and we can apply Lemma 2.

Let us construct the system possessing pseudopotential defined by (2.12), (2.14), (2.15) explicitly in terms of three linear independent solutions of the linear system (2.13).

Proposition 2. Let

$$\phi(\zeta) = \alpha_0 + \alpha_1\zeta + ... + \alpha_n\zeta^n,$$
$$\phi_1(\zeta) = \beta_0 + \beta_1\zeta + ... + \beta_n\zeta^n,$$
$$\phi_2(\zeta) = \gamma_0 + \gamma_1\zeta + ... + \gamma_n\zeta^n.$$

Then the system with pseudopotential defined by (2.12), (2.14), (2.15) can be written in the form

$$\sum_{i,j,k} \frac{u_i^{j+l}}{(u_i-u_1)...\hat{i}...(u_i-u_n)}((\gamma_j\alpha_k - \gamma_k\alpha_j)u_{it} + (\alpha_j\beta_k - \alpha_k\beta_j)u_{ix} + (\beta_j\gamma_k - \beta_k\gamma_j)u_{iy}) = 0. \quad (2.18)$$

Here summation is made by i, j, k subject to the constrains $1 \le i \le n$, $0 \le j \le l < k \le n$ with fixed l. For each $l = 1, ..., n$ we have an equation.

Proof. Substituting (2.16) into (1.10) we obtain

$$\sum_{i=1}^{n} \nu_i(\zeta)u_{it} + \sum_{i=1}^{n} \mu_i(\zeta)u_{ix} + \sum_{i=1}^{n} \vartheta_i(\zeta)u_{iy} = 0. \quad (2.19)$$

Calculating polynomials $\nu_i(\zeta), \mu_i(\zeta), \vartheta_i(\zeta)$ in terms of coefficients of polynomials $\phi(\zeta), \phi_1(\zeta), \phi_2(\zeta)$ and equating to zero coefficients at each powers of ζ in (2.19) we obtain (2.18).

Remark 1. Let $\phi_1, ..., \phi_{n+1}$ be linear independent solutions of the linear system (2.13). We can construct a system of the type (2.18) for each triplet of these solutions. It is clear that all these systems are compatible. Therefore, our systems have a lot of infinitesimal symmetries of hydrodynamic type.

Let us describe our pseudopotential written in the form (1.5). One can derive differential equations for the functions f, g from (2.12), (2.14) and (2.15).

Define a function $q(\xi, u_1, ..., u_n)$ as a solution of the following system of PDEs

$$q_\xi = \frac{q^{s_1}(q-1)^{s_2}(q-u_1)^{s_3}...(q-u_n)^{s_{n+2}}}{\phi(q)}, \quad (2.20)$$

$$q_{u_i} = \frac{\phi(u_i)}{\phi(q)} \cdot \frac{q(q-1)(q-u_1)...\hat{i}...(q-u_n)}{u_i(u_i-1)(u_i-u_1)...\hat{i}...(u_i-u_n)}, \quad i = 1, ..., n.$$

The system (2.20) is in involution iff the polynomial ϕ satisfies the linear system (2.13).

Let ϕ, ϕ_1 and ϕ_2 be three linear independent solutions of the system (2.13). Define functions $f(\xi, u_1, ..., u_n)$ and $g(\xi, u_1, ..., u_n)$ as a solution of the following system of PDEs

$$f_\xi = \frac{\phi_1(q)}{\phi(q)}, \qquad g_\xi = \frac{\phi_2(q)}{\phi(q)},$$

$$f_{u_i} = -\frac{\mu_i(q)}{\phi(q)} q^{1-s_1}(q-1)^{1-s_2}(q-u_1)^{1-s_3}...(q-u_n)^{1-s_{n+2}}, \tag{2.21}$$

$$g_{u_i} = \frac{\nu_i(q)}{\phi(q)} q^{1-s_1}(q-1)^{1-s_2}(q-u_1)^{1-s_3}...(q-u_n)^{1-s_{n+2}}, \quad i = 1, .., n.$$

Here μ_i, ν_i are defined by (2.17). It can be checked straightforwardly that the system (2.21) is in involution.

Proposition 3. If the functions f, g are defined by (2.21), then the system (1.5) is a pseudopotential for the system (2.18).

Proof. The formulas (2.21) imply

$$f_\xi g_{u_i} - g_\xi f_{u_i} = -\frac{\vartheta_i(q)}{\phi(q)} q^{1-s_1}(q-1)^{1-s_2}(q-u_1)^{1-s_3}...(q-u_n)^{1-s_{n+2}} \tag{2.22}$$

where ϑ_i is defined by (2.17). Therefore, the space of functions $\{f_{u_i}, g_{u_i}, f_\xi g_{u_i} - g_\xi f_{u_i}; i = 1, ..., n\}$ in the variable ξ is isomorphic to the space of polynomials in q of degree less or equal to $n-1$. This space is n-dimensional and we can apply Lemma 1. Let $\phi(q) = \alpha_0 + \alpha_1 q + ... + \alpha_n q^n$, $\phi_1(q) = \beta_0 + \beta_1 q + ... + \beta_n q^n$ and $\phi_2(q) = \gamma_0 + \gamma_1 q + ... + \gamma_n q^n$. Substituting (2.21), (2.22) into (1.6) we obtain (2.19). Calculating polynomials $\nu_i(q), \mu_i(q), \vartheta_i(q)$ in terms of coefficients of polynomials $\phi(q), \phi_1(q), \phi_2(q)$ and equating to zero coefficients at each powers of q in (2.19) we obtain (2.18).

Remark 2. Let $s_3 = ... = s_{n+2} = 1$ and $\phi(q) = (q-u_1)...(q-u_n)$. Then the system (2.18) coincides with a system from the paper [10] (see Example 4 in [10]).

3 The case $n = 2$

In this case each system (1.3) can be written in the form

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v_x \\ w_x \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} v_y \\ w_y \end{pmatrix} = 0 \tag{3.23}$$

in some suitable coordinates v, w. These systems were intensively studied in the paper [2]. In particular, it was proven in this paper that the system (3.23) is integrable iff the functions a, b, p, q, r, s satisfy the following system of PDEs

Equations for a:

$$
\begin{aligned}
a_{vv} &= \frac{qa_vb_v + 2qa_v^2 + (s-p)a_va_w - ra_w^2}{(a-b)q} + \frac{a_vr_v}{r} + \frac{2a_vp_w - a_wp_v}{q}, \\
a_{vw} &= a_v\frac{a_w + b_w}{a-b} + a_v(\frac{q_w}{q} + \frac{r_w}{r}), \\
a_{ww} &= \frac{qa_vb_v + (s-p)a_vb_w + ra_w^2}{(a-b)r} + \frac{a_vs_w}{r} + \frac{a_wq_w}{q}.
\end{aligned}
\tag{3.24}
$$

Equations for b:

$$
\begin{aligned}
b_{vv} &= \frac{ra_wb_w + (p-s)a_vb_w + qb_v^2}{(b-a)q} + \frac{b_wp_v}{q} + \frac{b_vr_v}{r}; \\
b_{vw} &= b_w\frac{a_v + b_v}{b-a} + b_w(\frac{q_v}{q} + \frac{r_v}{r}), \\
b_{ww} &= \frac{ra_wb_w + 2rb_w^2 + (p-s)b_vb_w - qb_v^2}{(b-a)r} + \frac{b_wq_w}{q} + \frac{2b_ws_v - b_vs_w}{r}.
\end{aligned}
\tag{3.25}
$$

Equations for p:

$$
\begin{aligned}
p_{vv} &= 2\frac{r(a_vb_w - a_wb_v) + (s-p)a_vb_v}{(a-b)^2} + \frac{r_vp_v}{r} + \frac{p_vp_w}{q} + \\
&\quad \frac{\frac{r}{q}(2q_va_w - 2a_vq_w + a_wp_w) - b_vp_v + 2r_va_w - 2a_v(s_v + p_v + r_w) + \frac{p-s}{q}(2p_va_w - a_vp_w)}{b-a} \\
p_{vw} &= 2(s-p)\frac{a_vb_w}{(a-b)^2} - \frac{b_wp_v + (2s_w + p_w)a_v}{b-a} + p_v\left(\frac{q_w}{q} + \frac{r_w}{r}\right), \\
p_{ww} &= 2\frac{q(a_wb_v - a_vb_w) + (s-p)a_wb_w}{(a-b)^2} + \frac{(p-s)b_wp_v - qb_vp_v - 2rs_wa_w - ra_wp_w}{(b-a)r} + \\
&\quad \frac{p_vs_w}{r} + \frac{q_wp_w}{q}.
\end{aligned}
\tag{3.26}
$$

Equations for s:

$$
\begin{aligned}
s_{vv} &= 2\frac{r(a_wb_v - a_vb_w) + (p-s)a_vb_v}{(a-b)^2} + \frac{(s-p)a_vs_w - ra_ws_w - 2qp_vb_v - qb_vs_v}{(a-b)q} + \\
&\quad \frac{p_vs_w}{q} + \frac{r_vs_v}{r}, \\
s_{vw} &= 2(p-s)\frac{a_vb_w}{(a-b)^2} - \frac{a_vs_w + (2p_v + s_v)b_w}{a-b} + s_w\left(\frac{q_v}{q} + \frac{r_v}{r}\right),
\end{aligned}
\tag{3.27}
$$

$$s_{ww} = 2\frac{q(a_v b_w - a_w b_v) + (p-s)a_w b_w}{(a-b)^2} + \frac{q_w s_w}{q} + \frac{s_v s_w}{r} + \frac{\frac{q}{r}(2r_w b_v - 2b_w r_v + b_v s_v) - a_w s_w + 2q_w b_v - 2b_w(p_w + s_w + q_v) + \frac{s-p}{r}(2s_w b_v - b_w s_v)}{a-b}.$$

Equations for q and r:

$$\begin{aligned}
qr_{ww} + rq_{ww} &= 2(p-s)\frac{(p-s)a_w b_w + q(a_v b_w - a_w b_v)}{(a-b)^2} + q\frac{r_v}{r}\frac{qb_v + (s-p)b_w}{a-b} + \\
&\quad (s-p)\frac{2a_w s_w + 2b_w p_w + b_w q_v}{a-b} + r\frac{(a_w - 2b_w)q_w}{a-b} + \\
&\quad q\frac{a_w r_w + b_v(2p_w + 2s_w + q_v) - 2b_w(r_w + p_v + s_v)}{a-b} + \\
&\quad \frac{r}{q}q_w^2 + \frac{q}{r}s_w r_v - q_w r_w + s_w(2p_w + q_v), \\
q_{vw} &= (s-p)\frac{qa_v b_v + (s-p)a_v b_w + ra_w b_w}{r(a-b)^2} + \frac{q_v q_w}{q} + \frac{p_v s_w}{r} + \\
&\quad \frac{a_v(rq_w + qr_w) + (s-p)(a_v s_w + b_w p_v) + ra_w s_w + qp_v b_v}{r(a-b)}, \\
r_{vw} &= (p-s)\frac{ra_w b_w + (p-s)a_v b_w + qa_v b_v}{q(a-b)^2} + \frac{r_v r_w}{r} + \frac{p_v s_w}{q} + \\
&\quad \frac{b_w(rq_v + qr_v) + (p-s)(a_v s_w + b_w p_v) + ra_w s_w + qp_v b_v}{q(b-a)}, \\
qr_{vv} + rq_{vv} &= 2(s-p)\frac{(s-p)a_v b_v + r(a_v b_w - a_w b_v)}{(a-b)^2} + r\frac{q_w}{q}\frac{ra_w + (p-s)a_v}{b-a} + \\
&\quad (p-s)\frac{2b_v p_v + 2a_v s_v + a_v r_w}{b-a} + q\frac{(b_v - 2a_v)r_v}{b-a} + \\
&\quad r\frac{b_v q_v + a_w(2s_v + 2p_v + r_w) - 2a_v(q_v + s_w + p_w)}{b-a} + \\
&\quad \frac{q}{r}r_v^2 + \frac{r}{q}p_v q_w - r_v q_v + p_v(2s_v + r_w);
\end{aligned} \tag{3.28}$$

Suppose that the system (3.23) possesses a pseudopotential of the form

$$\psi_t = f(\psi_y,\ v,\ w), \quad \psi_x = g(\psi_y,\ v,\ w). \tag{3.29}$$

Another remarkable result of the paper [2] is the following

Theorem. The class of two-component (2+1)-dimensional systems of hydrodynamic type possessing infinitely many hydrodynamic reductions coincides with the class of systems possessing a pseudopotential of the form (3.29).

Recall the prove of this theorem (see [2] for details). Writing out the consistency condition $\psi_{tx} = \psi_{xt}$, expressing $v_t,\ w_t$ by virtue of (3.23) and equating to zero coefficients at v_x, v_y, w_x, w_y, one arrives at the following expressions for the first derivatives f_v, f_w, f_ξ and g_ξ (we adopt the notation $\xi \equiv \psi_y$):

$$f_v = -a\ g_v, \quad f_w = -b\ g_w,$$
$$f_\xi = \frac{b\left(p + r\frac{g_w}{g_v}\right) - a\left(s + q\frac{g_v}{g_w}\right)}{a-b}, \tag{3.30}$$

and

$$g_\xi = \frac{s + q\frac{g_v}{g_w} - p - r\frac{g_w}{g_v}}{a-b}. \tag{3.31}$$

The consistency conditions of the equations (3.30) imply the following expressions for the second partial derivatives g_{vw}, g_{vv}, g_{ww}:

$$g_{vw} = \frac{a_w}{b-a}\ g_v + \frac{b_v}{a-b}\ g_w,$$
$$g_{vv} = \frac{g_v[g_w^2(r(b_v-a_v)+(a-b)r_v)+g_vg_w((a-b)p_v+(s-p)a_v-ra_w)+qa_vg_v^2]}{(a-b)rg_w^2}, \tag{3.32}$$
$$g_{ww} = \frac{g_w[g_v^2(q(a_w-b_w)+(b-a)q_w)+g_vg_w((b-a)s_w+(p-s)b_w-qb_v)+rb_wg_w^2]}{(b-a)qg_v^2}.$$

The compatibility conditions of the equations (3.31), (3.32) for g, namely, the conditions $g_{\xi vv} = g_{vv\xi},\ g_{\xi vw} = g_{vw\xi}$, etc., are of the form $P(g_v, g_w) = 0$, where P denotes a rational expression in $g_v,\ g_w$ whose coefficients are functions of a, b, p, q, r, s and their partial derivatives up to the second order. Equating all these expressions to zero (they are required to be zero identically in g_v, g_w), one obtains the set of conditions which are necessary and sufficient for the existence of a pseudopotential of the form (3.29). It turns out that these conditions identically coincide with the integrability conditions (3.24) - (3.28). Thus, any system satisfying the integrability conditions (3.24) - (3.28) possesses a pseudopotential of the form (3.29).

Based on these results of [2] we can prove the following

Proposition 4. If a system (3.23) corresponds to a general solution of (3.24) - (3.28), then in suitable coordinates it is equivalent to some system of the form (2.18) (in the case $n = 2$) and possesses a pseudopotential of the form (2.21).

Proof. Let us study the system (3.30)-(3.32). We denote $q = \frac{g_w}{g_v}$. Calculation shows that

$$\frac{q_{\xi\xi}}{q_\xi^2} = -\frac{1}{q} + \frac{P_4(q)}{P_5(q)}$$

where P_4, P_5 are polynomials in q of degree 4 and 5 correspondingly. Moreover, P_5 has distinct roots for general solution of the system (3.24) - (3.28). Write

$$\frac{q_{\xi\xi}}{q_\xi^2} = -\frac{1}{q} + \sum_{1\le i\le 5} \frac{s_i}{q - \lambda_i}$$

where s_i, λ_i are some functions in v, w. Calculation shows that $\sum_{1\leq i\leq 5} s_i = \lim_{q\to\infty} \frac{qP_4(q)}{P_5(q)} = 4$. After integration we get

$$q_\xi = \frac{C}{q} \prod_{1\leq i\leq 5} (q-\lambda_i)^{s_i}.$$

Note also that

$$f_\xi = \frac{P_2(q)}{q}, \quad g_\xi = \frac{Q_2(q)}{q}$$

where P_2, Q_2 are quadratic polynomials in q and

$$q_v = \frac{S_2(q)}{q}, \quad q_w = qG_2(q)$$

for some quadratic polynomials S_2, G_2. Let us make a substitution of the form

$$q \to \frac{\alpha q+\beta}{\gamma q+\delta} \tag{3.33}$$

where $\alpha, \beta, \gamma, \delta$ are functions in v, w. After that we obtain

$$\begin{gathered} q_\xi = \frac{1}{\phi(q)} \prod_{1\leq i\leq 5} (q-\rho_i)^{s_i}, \\ q_v = \frac{S_4(q)}{\phi(q)}, q_w = \frac{G_4(q)}{\phi(q)} \\ f_\xi = \frac{\phi_1(q)}{\phi(q)}, g_\xi = \frac{\phi_2(q)}{\phi(q)} \end{gathered} \tag{3.34}$$

where $\phi(q)$, $\phi_1(q)$, $\phi_2(q)$ are quadratic polynomials in q and S_4, G_4 are polynomials of degree 4.

Note that after appropriate substitution (3.33) we can assume $\rho_1 = 0$, $\rho_2 = 1$ and $\rho_5 = \infty$. Moreover, after change of variables $v = v(u_1, u_2)$, $w = w(u_1, u_2)$ we can assume $\rho_3 = u_1$, $\rho_4 = u_2$. Compatibility conditions for the system (3.34) with these assumptions imply that this system has a form (2.20) with $n = 2$ where ϕ satisfies (2.13). In particular, $s_1, ..., s_5$ must be constant. Moreover, coefficients of ϕ, ϕ_1, ϕ_2 must satisfy the same system of linear equations. Indeed, we can swap ϕ, ϕ_1, ϕ_2 by exchanging the role of t, x, y in (3.29). After that it is easy to find f_{u_i}, g_{u_i} in the form (2.21).

4 Elliptic case

Fix a complex parameter τ such that $\mathrm{Im}\tau > 0$. Define a function $\theta(z)$ by

$$\theta(z) = \sum_{m\in\mathbb{Z}} (-1)^m e^{-2\pi i(mz+\frac{m(m-1)}{2}\tau)}.$$

It is clear that $\theta(z)$ is the entire function satisfying the following relations:

$$\theta(z+1)=\theta(z), \quad \theta(z+\tau)=-e^{-2\pi i z}\theta(z).$$

Moreover, each entire function in one variable satisfying these relations is proportional to $\theta(z)$. We have also $\theta(-z)=-e^{-2\pi i z}\theta(z)$ and the only zero of the function $\theta(z)$ modulo 1 and τ is $z=0$.

Let $\Theta_{n,c}(\tau)$ be the space of the entire functions in one variable satisfying the following relations:

$$f(z+1)=f(z), \quad f(z+\tau)=(-1)^n e^{-2\pi i(nz-c)}f(z).$$

Here $n\in\mathbb{N}$ and $c\in\mathbb{C}$. It is known that $\dim\Theta_{n,c}(\tau)=n$, every function $f\in\Theta_{n,c}(\tau)$ has exactly n zeros modulo 1 and τ (counting according to their multiplicities), and the sum of these zeros is equal to c modulo 1 and τ. We have $\theta(z)\in\Theta_{1,0}(\tau)$.

Define a function $F(\zeta,u_1,\dots,u_n)$ as a solution of the following systems of PDEs

$$F_\zeta=\frac{\phi(\zeta)}{\theta(\zeta-u_1)...\theta(\zeta-u_n)},$$

$$F_{u_i}=-\frac{1}{\theta(u_i-u_1)...\hat{i}...\theta(u_i-u_n)}\cdot\frac{\theta(\zeta-u_i+\eta)}{\theta(\eta)\theta(\zeta-u_i)}\cdot\phi(u_i), \quad i=1,...,n. \tag{4.35}$$

Here η is a constant and $\phi\in\Theta_{n,u_1+\ldots+u_n-\eta}(\tau)$ as a function in ζ. This means that $\phi(\zeta)$ is the entire function in ζ and

$$\phi(\zeta+1)=\phi(\zeta), \quad \phi(\zeta+\tau)=(-1)^n e^{-2\pi i(n\zeta-u_1-\ldots-u_n+\eta)}\phi(\zeta). \tag{4.36}$$

We assume that η is nonzero modulo 1 and τ. The system (4.35) is in involution iff the function ϕ satisfies the following system of PDEs

$$\phi_{u_i}(\zeta)=\phi(u_i)\frac{\theta(\zeta-u_1)...\hat{i}...\theta(\zeta-u_n)}{\theta(u_i-u_1)...\hat{i}...\theta(u_i-u_n)}\cdot\frac{\theta(\zeta-u_i+\eta)}{\theta(\eta)}\times \tag{4.37}$$

$$\left(\frac{\theta'(\zeta-u_i)}{\theta(\zeta-u_i)}-\frac{\theta'(\zeta-u_i+\eta)}{\theta(\zeta-u_i+\eta)}\right)-\frac{\theta'(\zeta-u_i)}{\theta(\zeta-u_i)}\phi(\zeta), \quad i=1,...,n.$$

It is clear that if $\phi\in\Theta_{n,u_1+\ldots+u_n-\eta}(\tau)$, then the equations (4.37) are compatible with (4.36) and the right hand side of (4.37) is an entire function in ζ. Therefore, (4.37) is a well-defined system of linear PDEs for coefficients of ϕ with respect to some basis in the n-dimensional space $\Theta_{n,u_1+\ldots+u_n-\eta}(\tau)$. It can be checked straightforwardly that this system is in involution. Therefore, there are n linear independent solutions. Moreover, this system can be solved explicitly. Namely, let

$$\psi_i(\zeta)=\theta(\zeta-u_1)...\hat{i}...\theta(\zeta-u_n)\cdot\theta(\zeta-u_i+\eta), \quad i=1,...,n. \tag{4.38}$$

It can be checked straightforwardly that these functions are linear independent solutions of the system (4.37).

Assume that $n \geq 3$. Let ϕ, ϕ_1 and ϕ_2 be three linear independent solutions of the system (4.37). We assume that $\phi, \phi_1, \phi_2 \in \Theta_{n,u_1+...+u_n-\eta}(\tau)$ as functions in ζ. Define functions $G(\zeta, u_1, \ldots, u_n)$ and $H(\zeta, u_1, \ldots, u_n)$ similarly to (4.35) by

$$G_\zeta = \frac{\phi_1(\zeta)}{\theta(\zeta - u_1)...\theta(\zeta - u_n)},$$

$$G_{u_i} = -\frac{1}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(\zeta - u_i + \eta)}{\theta(\eta)\theta(\zeta - u_i)} \cdot \phi_1(u_i), \quad i = 1, ..., n \tag{4.39}$$

for the function G and

$$H_\zeta = \frac{\phi_2(\zeta)}{\theta(\zeta - u_1)...\theta(\zeta - u_n)},$$

$$H_{u_i} = -\frac{1}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(\zeta - u_i + \eta)}{\theta(\eta)\theta(\zeta - u_i)} \cdot \phi_2(u_i), \quad i = 1, ..., n \tag{4.40}$$

for the function H.

Proposition 5. If the functions F, G, H are defined by (4.35), (4.39) and (4.40), then the system (1.9) defines a pseudopotential for some system of the form (1.11) with $m = n$.

Proof. Equations (4.35), (4.39), (4.40) imply

$$\begin{aligned} H_\zeta G_{u_i} - G_\zeta H_{u_i} &= \frac{\vartheta_i(\zeta)}{\theta(\zeta - u_1)...\theta(\zeta - u_n)}, \\ F_\zeta H_{u_i} - H_\zeta F_{u_i} &= \frac{\nu_i(\zeta)}{\theta(\zeta - u_1)...\theta(\zeta - u_n)}, \\ G_\zeta F_{u_i} - F_\zeta G_{u_i} &= \frac{\mu_i(\zeta)}{\theta(\zeta - u_1)...\theta(\zeta - u_n)} \end{aligned} \tag{4.41}$$

where functions $\vartheta_i(\zeta), \nu(\zeta), \mu(\zeta)$ are defined by

$$\begin{aligned} \vartheta_i(\zeta) &= \frac{1}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(\zeta - u_i + \eta)}{\theta(\eta)\theta(\zeta - u_i)} \cdot (\phi_2(u_i)\phi_1(\zeta) - \phi_1(u_i)\phi_2(\zeta)), \\ \nu_i(\zeta) &= \frac{1}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(\zeta - u_i + \eta)}{\theta(\eta)\theta(\zeta - u_i)} \cdot (\phi(u_i)\phi_2(\zeta) - \phi_2(u_i)\phi(\zeta)), \\ \mu_i(\zeta) &= \frac{1}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(\zeta - u_i + \eta)}{\theta(\eta)\theta(\zeta - u_i)} \cdot (\phi_1(u_i)\phi(\zeta) - \phi(u_i)\phi_1(\zeta)). \end{aligned} \tag{4.42}$$

It is clear that $\vartheta_i(\zeta)$, $\nu_i(\zeta)$, $\mu_i(\zeta) \in \Theta_{n,u_1+...+u_n-2\eta}(\tau)$. Therefore, the space of functions $\{H_\zeta G_{u_i} - G_\zeta H_{u_i}, F_\zeta H_{u_i} - H_\zeta F_{u_i}, G_\zeta F_{u_i} - F_\zeta G_{u_i}; i = 1, ..., n\}$ in the variable ζ is isomorphic to the space $\Theta_{n,u_1+...+u_n-2\eta}(\tau)$. This space is n-dimensional and we can apply Lemma 2.

Let us construct the system possessing pseudopotential defined by (4.35), (4.39), (4.40) explicitly.

Proposition 6. Let

$$\phi(\zeta) = \sum_{i=1}^{n} \alpha_i \psi_i(\zeta), \quad \phi_1(\zeta) = \sum_{i=1}^{n} \beta_i \psi_i(\zeta), \quad \phi_2(\zeta) = \sum_{i=1}^{n} \gamma_i \psi_i(\zeta)$$

where $\psi_i(\zeta)$ are given by (4.38) and $\alpha_i, \beta_i, \gamma_i$ are constants. Then the system with pseudopotential defined by (4.35), (4.39), (4.40) can be written in the form

$$\sum_i \frac{\theta(u_j - u_i + \eta)}{\theta(u_j - u_i)}((\gamma_i\alpha_j - \gamma_j\alpha_i)(u_{it} - u_{jt}) + (\alpha_i\beta_j - \alpha_j\beta_i)(u_{ix} - u_{jx}) + (\beta_i\gamma_j - \beta_j\gamma_i)(u_{iy} - u_{jy})) = 0. \tag{4.43}$$

Here summation is made by i subject to the constrains $1 \leq i \leq n$, $i \neq j$ with fixed j. For each $j = 1, ..., n$ we have an equation.

Proof. Substituting (4.41) into (1.10) we obtain

$$\sum_{i=1}^{n} \nu_i(\zeta) u_{it} + \sum_{i=1}^{n} \mu_i(\zeta) u_{ix} + \sum_{i=1}^{n} \vartheta_i(\zeta) u_{iy} = 0. \tag{4.44}$$

Calculating functions $\nu_i(\zeta), \mu_i(\zeta), \vartheta_i(\zeta)$ in terms of functions $\phi(\zeta), \phi_1(\zeta), \phi_2(\zeta)$ and evaluating (4.44) at $\zeta = u_j$ we obtain (4.43).

Remark 3. If $n = 3$, then the system (4.43) reduces to the trivial system

$$u_{1t} = u_{3t}, \quad u_{2x} = u_{1x}, \quad u_{2y} = u_{3y}.$$

Therefore, one can assume $n > 3$.

Remark 4. The system (4.43) is invariant under translation $u_i \to u_i + v$ for an arbitrary function v. Therefore, one can reduce the number of unknown functions in the system setting $u_n = 0$.

Remark 5. The system (4.43) is written in the form (1.11) and can not be written in the form (1.3). In particular, it does not belong to the class of the systems studied in the paper [3].

Let us describe our pseudopotential written in the form (1.5). One can derive differential equations for the functions f, g from (4.35), (4.39) and (4.40).

Define a function $q(\xi, u_1, ..., u_n)$ as a solution of the following system of PDEs

$$q_\xi = \frac{\theta(q - u_1)...\theta(q - u_n)}{\phi(q)}, \tag{4.45}$$

$$q_{u_i} = \frac{\phi(u_i)}{\phi(q)} \cdot \frac{\theta(q - u_1)...\hat{i}...\theta(q - u_n)}{\theta(u_i - u_1)...\hat{i}...\theta(u_i - u_n)} \cdot \frac{\theta(q - u_i + \eta)}{\theta(\eta)}, \quad i = 1, ..., n.$$

The system (4.45) is in involution iff the function $\phi \in \Theta_{n,u_1+...+u_n-\eta}(\tau)$ satisfies (4.37).

Let ϕ, ϕ_1, $\phi_2 \in \Theta_{n,u_1+...+u_n-\eta}(\tau)$ be three linear independent solutions of the system (4.37). Define functions $f(\xi, u_1, ..., u_n)$ and $g(\xi, u_1, ..., u_n)$ as a solution of the following system of PDEs

$$f_\xi = \frac{\phi_1(q)}{\phi(q)}, \qquad g_\xi = \frac{\phi_2(q)}{\phi(q)},$$

$$f_{u_i} = -\frac{\mu_i(q)}{\phi(q)}, \quad g_{u_i} = \frac{\nu_i(q)}{\phi(q)}, \quad i = 1, .., n. \tag{4.46}$$

Here μ_i, ν_i are defined by (4.42). It can be checked straightforwardly that the system (4.46) is in involution.

Proposition 7. If the functions f, g are defined by (4.46), then the system (1.5) is a pseudopotential for some system of the form (1.11).

Proof. The formulas (4.46) imply

$$f_\xi g_{u_i} - g_\xi f_{u_i} = -\frac{\vartheta_i(q)}{\phi(q)} \tag{4.47}$$

where ϑ_i is defined by (4.42). Therefore, the space of functions $\{f_{u_i}, g_{u_i}, f_\xi g_{u_i} - g_\xi f_{u_i}; i = 1, ..., n\}$ in the variable ξ is isomorphic to the space $\Theta_{n,u_1+...+u_n-2\eta}(\tau)$ in the variable q. This space is n-dimensional and we can apply Lemma 2.

5 Discussion

In this paper we suggest a construction of n-component (2+1)-dimensional hydrodynamic type systems possessing pseudopotential. Let us outline some features of this construction. It is clear from (1.7), (1.8) that there exists a polynomial $S(u, v)$ of degree n such that $S(f_\xi, g_\xi) = 0$. If a curve $K = \{(u, v); S(u, v) = 0\}$ is rational, then it is natural to write f_ξ, g_ξ in the form (2.21) where ϕ, ϕ_1, ϕ_2 are polynomials of degree n. It turns out that the coefficients of each polynomial satisfy the same system of linear PDEs. A similar construction exists if the curve K is elliptic, but in this case we get an overdetermined integrable system with n equations for $n-1$ unknowns. The natural conjecture is that there exists a similar construction in the case $g > 1$ where g is the genus of K and the number of equations in the corresponding integrable system should be g plus the number of unknowns. It would be interesting to verify this conjecture and find these systems explicitly as well as their degenerations. Some (or may be even all in the case $g > 0$) of these systems could be reductions of the universal Whitham hierarchy [7].

Our construction provides a general solution to the classification problem of two-component integrable hydrodynamic type systems. The full classification of these systems as well as their detailed study will be the subject of a separate paper.

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